Mathematical Logic

Part Three

Recap from Last Time

What is First-Order Logic?

First-order logic is a logical system for reasoning about properties of objects.

Augments the logical connectives from propositional logic with

- ***predicates*** that describe properties of objects,
- ***functions*** that map objects to one another, and
- ***quantifiers*** that allow us to reason about many objects at once.

Some muggle is intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

"For any natural number n,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

$\forall$ is the ***universal quantifier*** and says "for any choice of *n*, the following is true."

“All A's are B's”

translates as

$$\forall x.\ (A(x) \rightarrow B(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (A(x) \to B(x))$$

If *x* is a counterexample, it *must* have property A but not have property *B*.

“Some *A* is a *B*”

translates as

$$\exists x.\ (A(x) \land B(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (A(x) \land B(x))$$

If *x* is an example, it *must* have property *A* on top of property *B*.

The Aristotelian Forms

"All *A*s are *B*s"

$\forall x. (A(x) \to B(x))$

"Some *A*s are *B*s"

$\exists x. (A(x) \land B(x))$

"No *A*s are *B*s"

$\forall x. (A(x) \to \neg B(x))$

"Some *A*s aren't *B*s"

$\exists x. (A(x) \land \neg B(x))$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- $Person(p)$, which states that p is a person, and
- $Loves(x, y)$, which states that x loves y,

write a sentence in first-order logic that means "every person loves someone else."

Every person loves someone else

Every person loves some other person

Every person p loves some other person

Every person p loves some other person

“All *A*s are *B*s”

$$\forall x.\ (A(x) \to B(x))$$

$\forall p. \ (Person(p) \rightarrow$

p loves some other person

$)$

"All *A*s are *B*s"

$\forall x. \ (A(x) \rightarrow B(x))$

$\forall p. \ (Person(p) \rightarrow$

p loves some other person

$)$

$\forall p. \ (Person(p) \rightarrow$

there is some other person that p loves

$)$

$\forall p. \ (Person(p) \to$

there is a person other than p that p loves

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where p loves q

$)$

$\forall p. \; (Person(p) \rightarrow$

there is a person q, other than p, where p loves q

$)$

"Some *A*s are *B*s"

$\exists x. \; (A(x) \land B(x))$

$\forall p.\ (Person(p) \rightarrow$

$\exists q.\ (Person(q) \land$ *, other than p, where*

p loves q

)

)

"Some *A*s are *B*s"

$\exists x.\ (A(x) \land B(x))$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q)\ \wedge$*, other than p, where*

p loves q

$\quad)$

$)$

$$\forall p.\ (Person(p) \rightarrow$$

$$\exists q.\ (Person(q) \land p \neq q \land$$

p loves q

$$)$$

$$)$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\qquad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad\qquad Loves(p, q) \\
&\qquad) \\
&)
\end{aligned}
$$

Using the predicates

- $Person(p)$, which states that p is a person, and
- $Loves(x, y)$, which states that x loves y,

write a sentence in first-order logic that means
"there is a person that everyone else loves."

There is a person that everyone else loves

There is a person p where everyone else loves p

There is a person p where everyone else loves p

"Some *A*s are *B*s"

$$\exists x.\ (A(x) \land B(x))$$

$\exists p. \ (Person(p) \wedge$

everyone else loves p

$)$

"Some *A*s are *B*s"

$\exists x. \ (A(x) \wedge B(x))$

$\exists p.\ (Person(p) \land$

everyone else loves p

$)$

$\exists p. \; (Person(p) \; \land$

every other person q loves p

$)$

$\exists p.\ (Person(p)\ \land$

every person q, other than p, loves p

$)$

$\exists p.\ (Person(p)\ \land$

every person q, other than p, loves p

$)$

"All *A*s are *B*s"

$\boldsymbol{\forall x.\ (A(x) \rightarrow B(x))}$

$\exists p.\ (Person(p)\ \land$

$\quad\forall q.\ (Person(q)\ \land\ p \neq q \rightarrow$

$\quad\quad$ *q loves p*

$\quad)$

$)$

"All *A*s are *B*s"

$\boldsymbol{\forall x.\ (A(x) \rightarrow B(x))}$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\qquad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\qquad\qquad \textit{q loves p} \\
&\qquad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\qquad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\qquad\qquad Loves(q, p) \\
&\qquad) \\
&)
\end{aligned}
$$

Combining Quantifiers

Most interesting statements in first-order logic require a combination of quantifiers.

Example: "Every person loves someone else"

For every person…	$\forall p.\ (Person(p) \rightarrow$
… there is another person …	$\exists q.\ (Person(q) \land p \neq q \land$
… they love	$Loves(p, q)$
	$)$
	$)$

Combining Quantifiers

Most interesting statements in first-order logic require a combination of quantifiers.

Example: "There is someone everyone else loves."

There is a person…	$\exists p. \; (Person(p) \land$
… that everyone else …	$\quad \forall q. \; (Person(q) \land p \neq q \rightarrow$
… loves.	$\qquad Loves(q, p)$
	$\quad)$
	$)$

For Comparison

For every person…

… there is another person …

… they love

$$\forall p.\ (Person(p) \to \\ \quad \exists q.\ (Person(q) \land p \neq q \land \\ \qquad Loves(p, q) \\ \quad) \\ )$$

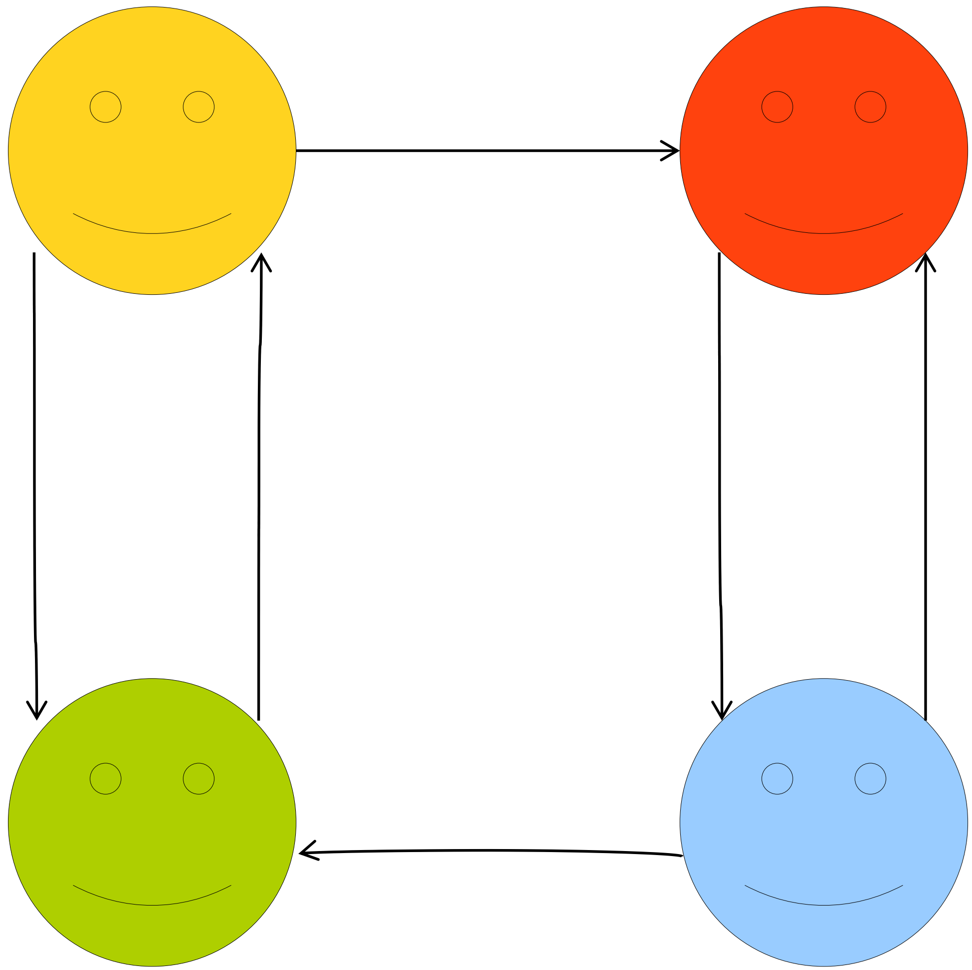
There is a person…

… that everyone else …

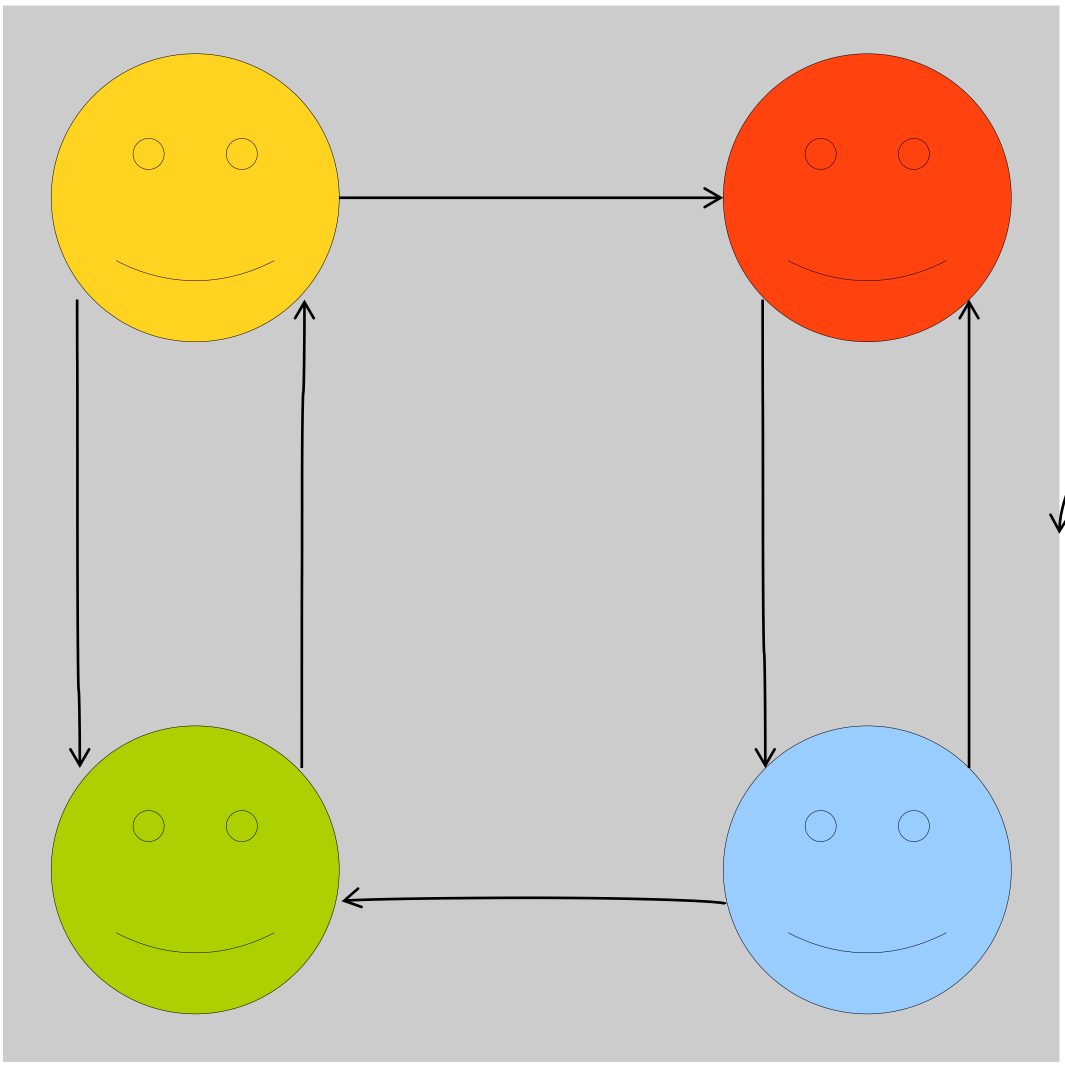
… loves.

$$\exists p.\ (Person(p) \land \\ \quad \forall q.\ (Person(q) \land p \neq q \to \\ \qquad Loves(q, p) \\ \quad) \\ )$$

Every Person Loves Someone Else

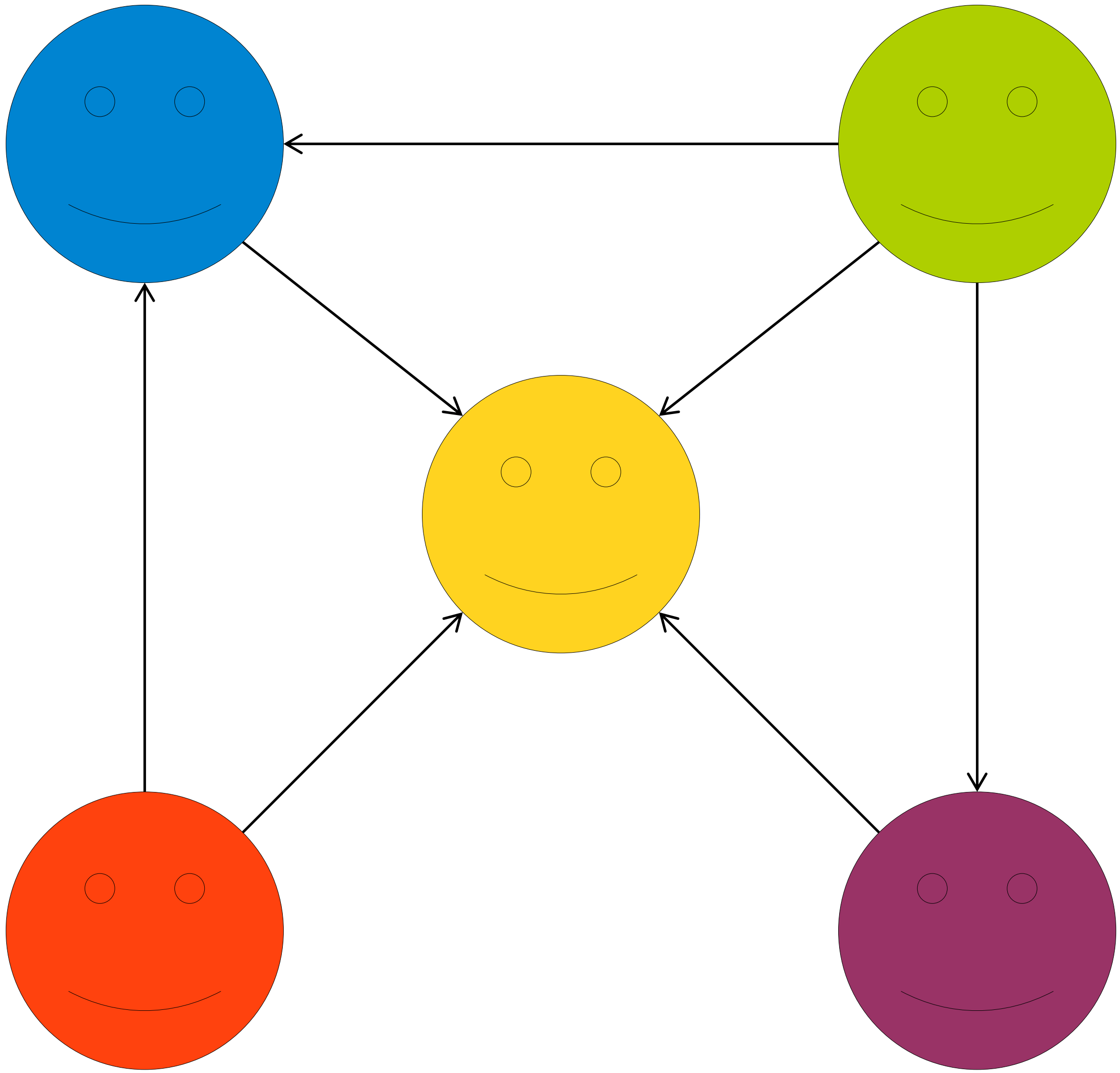


Every Person Loves Someone Else

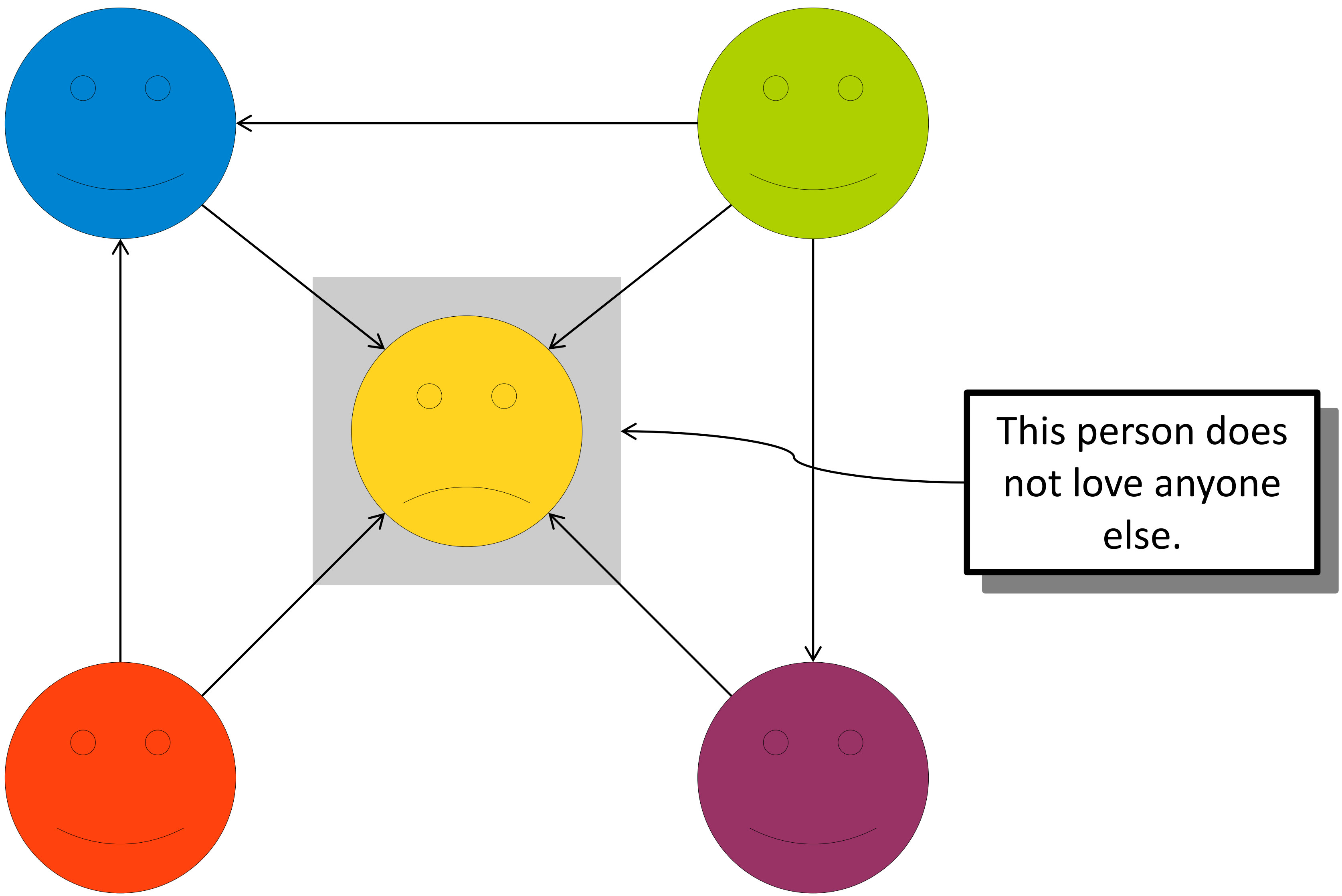


No one here is universally loved.

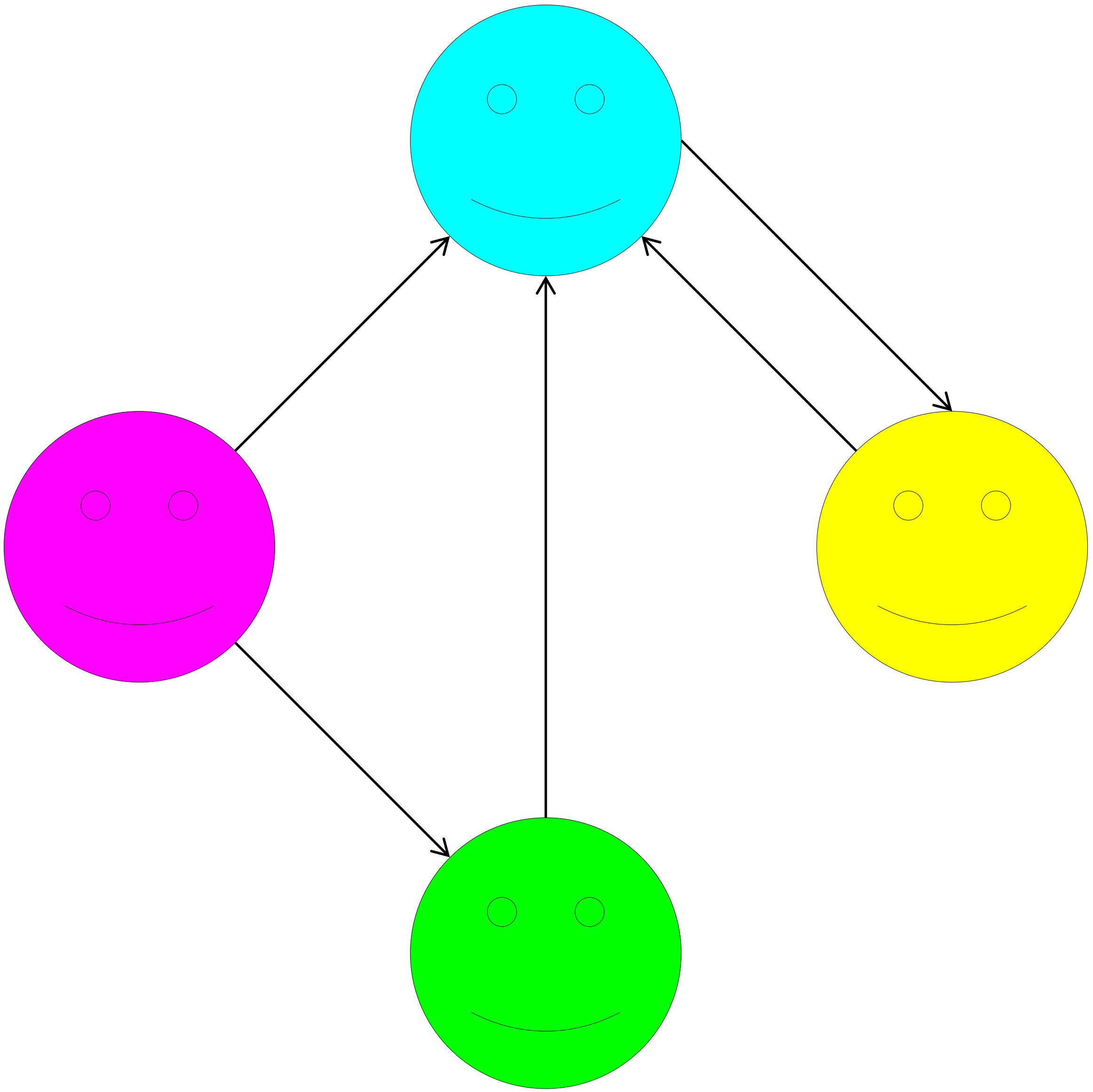
There is Someone Everyone Else Loves



There is Someone Everyone Else Loves



Every Person Loves Someone Else ***and*** There is Someone Everyone Else Loves



For every person… $\forall p.\ (Person(p) \to$

… there is another person … $\exists q.\ (Person(q) \land p \neq q \land$

… they love $Loves(p, q)$

$)$

$)$

and $\land$

There is a person… $\exists p.\ (Person(p) \land$

… that everyone else … $\forall q.\ (Person(q) \land p \neq q \to$

… loves. $Loves(q, p)$

$)$

$)$

Quantifier Ordering

The statement

$$\forall x.\ \exists y.\ P(x, y)$$

means “for any choice of x, there's some choice of y where $P(x, y)$ is true.”

The choice of y can be different every time and can depend on x.

Quantifier Ordering

The statement

$$\exists x.\ \forall y.\ P(x, y)$$

means “there is some x where for any choice of y, we get that $P(x, y)$ is true.”

Since the inner part has to work for any choice of y, this places a lot of constraints on what x can be.

Order matters when mixing existential and universal quantifiers!

Time-Out for Announcements!

Problem Set Two

- Problem Set Two went out last Friday.
- There is no checkpoint assignment.
- All problems are due this Thursday at 11:59PM.
- We have some reading recommendations for this problem set.
- Check out the ***Guide to Logic Translations*** for more on how to convert from English to FOL.
- Check out the ***Guide to Negations*** for information about how to negate formulas.
- Check out the ***First-Order Translation Checklist*** for details on how to check your work.

Problem Set One Solutions

- We’ve just released solutions to Problem Set One.
- ***You need to read over these solutions as soon as possible***.
- Why?
- Each question is there for a reason. We’ve described what it is that we hoped you would have learned when solving those problems.
- There are lots of different ways of solving these problems. Comparing what you did against our solutions, which are just one possible set of solutions, can help introduce new techniques.

Back to CS103!

Set Translations

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

First-order logic doesn't have set operators or symbols "built in." If we only have the predicates given above, how might we describe this?

The empty set exists.

There is some set S that is empty.

$\exists S. \; (Set(S) \;\wedge$

S is empty.

$)$

$\exists S.\ (Set(S)\ \land$

there are no elements in S

$)$

$\exists S.\ (Set(S)\ \land$

$\neg$*there is an element in S*

$)$

$\exists S.\ (Set(S)\ \land$

$\neg$*there is an element x in S*

$)$

$$\exists S.\ (Set(S)\ \land$$

$$\neg\exists x.\ x \in S$$

$$)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$\exists S.\ (Set(S) \land$

there are no elements in S

$)$

$$\exists S.\ (Set(S)\ \land\ \neg\exists x.\ x \in S)$$

$\exists S.\ (Set(S)\ \land$

every object does not belong to S

$)$

$\exists S. \, (Set(S) \land \neg \exists x. \, x \in S)$

$\exists S. \, (Set(S) \land$

every object x does not belong to S

$)$

$$\exists S. \ (Set(S) \ \land \ \neg\exists x. \ x \in S)$$

$$\exists S. \ (Set(S) \ \land$$

$$\forall x. \ x \notin S$$

$$)$$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “two sets are equal if and only if they contain the same elements.”

Two sets are equal if and only if they have the same elements.

Any two sets are equal if and only if they have the same elements.

Any two sets S and T are equal if and only if they have the same elements.

$\forall S. \ (Set(S) \rightarrow$

$\forall T. \ (Set(T) \rightarrow$

S and T are equal if and only if they have the same elements.

$)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\forall T.\ (Set(T) \rightarrow$

$(S = T$ *if and only if they have the same elements.*$)$

$)$

$)$

$$\forall S.\ (Set(S) \to$$

$$\forall T.\ (Set(T) \to$$

$$(S = T \leftrightarrow \textit{they have the same elements.})$$

$$)$$

$$)$$

$\forall S. \ (Set(S) \rightarrow$

$\quad \forall T. \ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *S and T have the same elements.*$)$

$\quad)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\forall T.\ (Set(T) \rightarrow$

$(S = T \leftrightarrow$ *every element of S is an element of T and*

vice-versa)

)

)

$\forall S.\ (Set(S) \rightarrow$

$\forall T.\ (Set(T) \rightarrow$

$(S = T \leftrightarrow$ *x is an element of S if and only if x is an*

element of T)

)

)

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\qquad \forall T.\ (Set(T) \rightarrow \\
&\qquad\qquad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\qquad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\qquad \forall T.\ (Set(T) \rightarrow \\
&\qquad\qquad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\qquad) \\
&)
\end{aligned}
$$

$$\forall S.\ (Set(S) \to$$

$$\forall T.\ (Set(T) \to$$

$$(S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T))$$

$$)$$

$$)$$

You sometimes see the universal quantifier pair with the ↔ connective. This is especially common when talking about sets because two sets are equal when they have precisely the same elements.

$$
\begin{aligned}
&\forall S.\ (Set(S) \to \\
&\qquad \forall T.\ (Set(T) \to \\
&\qquad\qquad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\qquad) \\
&)
\end{aligned}
$$

Restricted Quantifiers

Quantifying Over Sets

The notation

$$\forall x \in S.\ P(x)$$

means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

The notation

$$\exists x \in S.\ P(x)$$

means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)

Quantifying Over Sets

The syntax

$$\forall x \in S.\ P(x)$$

$$\exists x \in S.\ P(x)$$

is allowed for quantifying over sets.

In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.

For example, don't do things like this:

⚠ $\forall x \text{ with } P(x).\ Q(x)$ ⚠

⚠ $\forall y \text{ such that } P(y) \land Q(y).\ R(y).$ ⚠

⚠ $\exists P(x).\ Q(x)$ ⚠

Let’s take a five minute break!

Expressing Uniqueness

Using the predicate

- *WayToFindOut*(*w*), which states that *w* is a way to find out,

write a sentence in first-order logic that means
“there is only one way to find out.”

There is only one way to find out.

Something is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing besides w is a way to find out

$\exists w.\ (WayToFindOut(w)\ \wedge$

nothing besides w is way to find out

$)$

$\exists w.\ (WayToFindOut(w)\ \wedge$

anything that isn't w isn't a way to find out

$)$

$\exists w.\ (WayToFindOut(w)\ \wedge$

any thing x that isn't w isn't a way to find out

$)$

$\exists w.\ (\mathit{WayToFindOut}(w)\ \land$

$\quad\quad \forall x.\ (x \neq w \rightarrow$ *x isn't a way to find out*$)$

$)$

$$
\exists w.\ (\mathit{WayToFindOut}(w)\ \land
$$

$$
\forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x))
$$

$$
)
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\qquad \forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\qquad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

Expressing Uniqueness

To express the idea that there is exactly one object with some property, we write that

- there exists at least one object with that property, and that
- there are no other objects with that property.

You sometimes see a special “uniqueness quantifier” used to express this:

$$\exists! x.\ P(x)$$

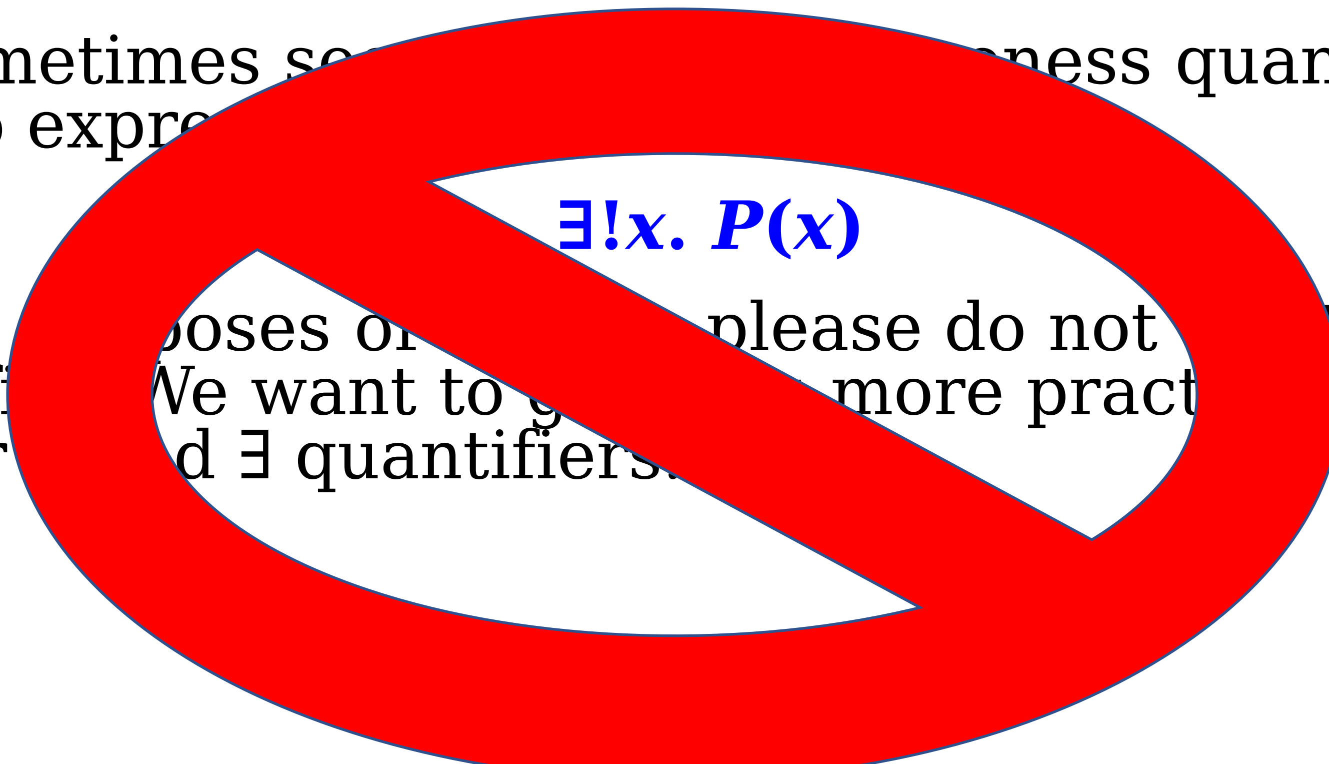
For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

Expressing Uniqueness

To express the idea that there is exactly one object with some property, we write that

- there exists at least one object with that property, and that
- there are no other objects with that property.

You sometimes se[illegible]ness quantifier” used to expre[illegible]



$$\exists! x.\ P(x)$$

For the [illegible]poses of [illegible] please do not [illegible] this quantifi[illegible] We want to g[illegible] more pract[illegible] using the regular [illegible]d ∃ quantifiers.

Mechanics: Negating Statements

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	For some choice of x, $\neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	**For some choice of x, $\neg P(x)$**
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	**$\exists x.\ \neg P(x)$**
$\exists x.\ P(x)$	For some choice of x, $P(x)$	**For any choice of x, $\neg P(x)$**
$\forall x.\ \neg P(x)$	**For any choice of x, $\neg P(x)$**	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	**For any choice of x, $\neg P(x)$**	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	**$\exists x.\ \neg P(x)$**
$\exists x.\ P(x)$	**For some choice of x, $P(x)$**	**$\forall x.\ \neg P(x)$**
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	**For some choice of x, $P(x)$**
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**For some choice of x, $P(x)$**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For any choice of *x*, *P*(*x*)**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	**For any choice of *x*, *P*(*x*)**

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For any choice of *x*, $P(x)$**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of *x*, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of *x*, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of *x*, $\neg P(x)$	$\forall x.\ P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

Negating First-Order Statements

Use the equivalences

$$\neg \forall x.\ A \quad \equiv \quad \exists x.\ \neg A$$

$$\neg \exists x.\ A \quad \equiv \quad \forall x.\ \neg A$$

to negate quantifiers.

Mechanically:

- Push the negation across the quantifier.
- Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$

$$\exists x.\ \neg\exists y.\ Loves(x, y)$$

$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \quad \equiv \quad p \to \neg q$$

$$\neg(p \to q) \quad \equiv \quad p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
- $\land$ is used in existentially-quantified statements.
- $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

What is the negation of the following statement, which says “there is a cute puppy”?

$$\boldsymbol{\exists x.\ (Puppy(x) \land Cute(x))}$$

We can obtain it as follows:

$$\neg\exists x.\ (Puppy(x) \land Cute(x))$$

$$\forall x.\ \neg(Puppy(x) \land Cute(x))$$

$$\forall x.\ (Puppy(x) \to \neg Cute(x))$$

This says “no puppy is cute.”

Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

(*"There is a set with no elements."*)

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$

$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$

$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$

$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

(*"Every set contains at least one element."*)

These two statements are *not* negations of one another. Can you explain why?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

("There is a set that doesn't contain anything")

$$\forall S.\ (Set(S) \land \exists x.\ (x \in S))$$

("Everything is a set that contains something")

Remember: ∀ usually goes with →, not ∧

Recap

Mixing Quantifiers

- Complex logic often requires multiple quantifiers.

Quantifier Ordering

- Changes the meaning.

Negating Quantifiers

- Keeping preferred connectives and maximally simplifying.

Next Time

Binary Relations

- How do we model connections between objects?

Equivalence Relations

- How do we model the idea that objects can be grouped into clusters?

First-Order Definitions

- Where does first-order logic come into all of this?

Proofs with Definitions

- How does first-order logic interact with proofs?